Construction of Compromise Values for Cooperative Games*

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Abstract

We explore a broad class of values for cooperative games in characteristic function form, known as *compromise values*. These values efficiently allocate payoffs by linearly combining well-specified upper and lower bounds on payoffs. We identify subclasses of games that admit non-trivial efficient allocations within the considered bounds, which we call *bound-balanced games*. Subsequently, we define the associated compromise value. We also provide an axiomatisation of this class of compromise values using a combination of the minimal-rights property and a variant of restricted proportionality.

We construct and axiomatise various well-known and new compromise values based on these methods, including the τ-, the χ-, the Gately, the CIS-, the PANSC-, the EANSC-, and the new KM-values. We conclude that this approach establishes a common foundation for a wide range of different values.

Keywords: Cooperative TU-game; compromise value; τ-value; CIS value; Gately value; KM-value; axiomatisation; bound balanced games.

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1 Introduction

In the theory of cooperative games with transferable utility (TU-games), the analysis of values has resulted in a large, prominent literature has been developed. A value on a particular class of TU-games assigns to every game in this class an efficient allocation of the proceeds of the grand coalition to the individual members of the population. Most prominent is the Shapley value (Shapley, 1953) and its variations, which is firmly rooted in well-accepted axiomatic characterisations. Other values are founded on selecting Core imputations, if the Core (Edgeworth, 1881; Gillies, 1959) is non-empty, such as the Nucleolus (Schmeidler, 1969). In this paper, we investigate an alternative class of values, known as *compromise values*.

A compromise value assigns the efficient balance of a given upper bound and lower bound to any TU-game for which the stated upper and lower bounds are well-defined. The main precedent of such a compromise value is the τ-value (Tijs, 1981), based on balancing the marginal contributions as an upper bound and the minimal rights payoffs as a lower bound. The τ-value is well-defined on the class of semi-balanced games (Tijs, 1981). Further analysis of the τ-value as a compromise value and its axiomatisation on the class of quasi-balanced games has been developed in Driessen and Tijs (1985); Tijs and Driessen (1987); Tijs (1987); Calvo et al. (1995); Bergantiños and Massó (1996); Bilbao et al. (2001) and González Díaz et al. (2005).

Although we restrict our definitions and analysis of compromise values to the realm of cooperative games in characteristic function form with transferable utility, Otten (1990); Borm et al. (1992) as well as Tijs and Otten (1993) extend the concept of compromise values to other realms such as non-transferable utility games and bargaining problems.

We investigate the broad class of compromise values that are based on so-called *bound pairs* composed of two functionals on the space of all TU-games that satisfy three regularity properties. These three regularity properties are naturally to be expected to be satisfied by functionals that represent upper and lower bounds on allocations in a TU-game. This general approach encompasses the construction method introduced in Sánchez-Soriano (2000), which imposes a more restrictive covariance property on the upper bound functional.

Our approach allows the identification of a rather broad class of compromise values including well-known values such as the τ-, χ-, Egalitarian Division, Gately, Centre of the Imputation Set (CIS), Proportional Allocation of Non-Separable Contributions (PANSC), and Equal Allocation of Non-Separable Contributions (EANSC) values. We also introduce a new compromise value, the Kikuta-Milnor (KM) value based on a lower bound function introduced by Kikuta (1980) and an upper bound functional considered by Milnor (1952).

Compromise values for a given bound pair are only well-defined on the subclass of bound-balanced TU-games corresponding to the TU-games that admit efficient allocations between the given upper and lower bounds. With reference to the τ-value introduced by Tijs (1981), this subclass is that of quasi- and semi-balanced games. We show that the KM-bound pair—based on the bound functionals introduced by Kikuta (1980) and Milnor (1952)—admits the complete space of TU-games as the corresponding KM-bound-balanced games. The other compromise values considered here usually refer to strict subclasses of the space of all TU-games.

We also consider an axiomatisation of any compromise value. The compromise value with respect to some bound pair is the unique allocation rule that—besides the standard efficiency property—satisfies two main properties: a form of covariance, known as the *Minimal Rights* property, and the *Restricted Proportionality* property.

Constructing bound pairs and their associated compromise values. We introduce two construction methods for compromise values based on a single, fixed functional. First, we consider constructing compromise values based on a lower bound functional that adheres to a regularity property. Specifically, the allocation functional that assigns each player the remaining value after paying all other players their lower bound forms a bound pair with the given lower bound functional. Consequently, the resulting compromise value is entirely determined by the chosen lower bound. We demonstrate that the Egalitarian Division rule, the CIS-value, and the EANSC-value are compromise values that can be constructed in this way using appropriately selected lower bound functionals for the specified construction method.

We show that these lower bound based compromise values can be axiomatised by the *Minimal Rights* property as well as an *Egalitarian Allocation* property that imposes the assignment of an equal share to all players for lower bound normalised games. The introduced Egalitarian Allocation property is more restrictive than the Restricted Proportionality property.

Second, we consider the construction of compromise values based on a given upper bound functional only. This construction method follows the framework set out by Tijs (1981) for the τ-value. This method identifies a minimal rights payoff vector based on the given upper bound on the payoffs in the TU-game. We show that if the upper bound functional is translation covariant, the constructed minimal rights payoff vector indeed defines an appropriate lower bound functional to form a bound pair with the given upper bound functional.

The considered covariance property on the upper bound functional is less stringent than the covariance property introduced by Sánchez-Soriano (2000). This paper considers the same construction method for a corresponding lower bound, which is the minimal rights payoff vector for the given upper bound. Therefore, our construction method generalizes the construction method introduced by Sánchez-Soriano (2000). We demonstrate that the τ-value and the CIS-value can be constructed using this method. This implies that the CIS-value can be constructed from its characteristic lower bound functional as well as its characteristic upper bound functional. This highlights the singular nature of the CIS-value as a value that can be constructed from both a lower bound as well as an upper bound.

Structure of the paper. The paper first discusses in Section 2 the necessary preliminaries including a discussion of the τ-value before the main concepts in the definition of compromise values are introduced.

Section 3 introduces well defined lower- and upper bound pairs, and applies the method of Tijs (1981) to define the value that assigns to every game on an appropriate subclass of TU-games the unique efficient allocation on the line segment between these two bounds. This section also extends the axiomatisation of the τ-value given in Tijs (1987) to the class of all those compromise values, and discusses the PANSC-, Gately and KM-values as examples of such compromise values that cannot be constructed in this way from appropriately chosen lower or upper bounds.

Section 4 introduces the construction of compromise values based on regular lower bound functionals and shows that the Egalitarian Division rule, the CIS-value and the EANSC-value can be constructed through this method. We provide an axiomatisation of these Lower Bound based Compromise (LBC) values that is a variant of the axiomatisation for compromise values in general.

Section 5 discusses the construction of compromise values based on translation covariant upper bound functionals. We show that the τ-value as well as the CIS-value are such Upper Bound based Compromise (UBC) values.

2 Preliminaries: Cooperative games and values

We first discuss the foundational concepts of cooperative games and solution concepts. Let $N = \{1, \dots, n\}$ be an arbitrary finite set of players and let $2^N = \{S \mid S \subseteq N\}$ be the corresponding set of all (player) coalitions in N. For ease of notation we usually refer to the singleton $\{i\}$ simply as i. Furthermore, we use the simplified notation $S - i = S \setminus \{i\}$ for any $S \in 2^N$ and $i \in S$ as well as $S + i = S \cup \{i\}$ for any $S \in 2^N$ and $i \in N \setminus S$.

A cooperative game with transferable utility—shortly referred to as a *cooperative game*, or simply as a *game*—on N is a function $v \colon 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$. A game assigns to every coalition a value or "worth" that this coalition can generate through the cooperation of its members. We refer to $v(S)$ as the *worth* of coalition $S \subseteq N$ in the game v.

The class of all cooperative games in the player set N is denoted by

$$\mathbb{V}^N = \left\{ v \,\middle|\, v \colon 2^N \to \mathbb{R} \text{ such that } v(\varnothing) = 0 \right\} \tag{1}$$

The class $\mathbb{V}^N$ forms an $(2^n - 1)$-dimensional linear real vector space.

The *dual* of a game $v \in \mathbb{V}^N$ is the game $v^* \in \mathbb{V}^N$ defined by $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$.

We use the natural ordering of the Euclidean space $\mathbb{V}^N$ to compare different cooperative games. In particular, we denote $v \leqslant w$ if and only if $v(S) \leqslant w(S)$ for all $S \neq \varnothing$; $v < w$ if and only if $v \leqslant w$ and $v \neq w$; and, finally, $v \ll w$ if and only if $v(S) < w(S)$ for all $S \neq \varnothing$.[1]

One can consider several bases of this linear vector space. For the study of compromise values the so-called standard base is the most useful. Formally, the *standard base* is given by the finite collection of games $\{b_S \mid S \subseteq N\} \subset \mathbb{V}^N$, for every coalition $S \subseteq N$ defined by[2]

$$b_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{if } T \neq S \end{cases}$$

For every player $i \in N$ let $v_i = v(\{i\})$ be their individually feasible worth in the game v. We refer

[1] We use the same notational convention to compare vectors in arbitrary Euclidean spaces.

[2] As such the standard base corresponds to the standard base of unit vectors in the Euclidean vector space $\mathbb{V}^N$.

to the game v as being *zero-normalised* if $v_i = 0$ for all $i \in N$. The collection of all zero-normalised games is denoted by $\mathbb{V}_0^N \subset \mathbb{V}^N$.

For any vector of payoffs $x \in \mathbb{R}^N$ we denote $x(S) = \sum_{i \in S} x_i$ for any coalition of players $S \in 2^N$ as the total assigned payoff to a certain coalition of players. As such, $x \in \mathbb{R}^N$ defines a corresponding trivial additive game $x \in \mathbb{V}^N$. Hence, the game $x \in \mathbb{V}^N$ assigns to every coalition the sum of the individual contributions of its members; there are no cooperative effects from bringing players together in that coalition.

For any cooperative game $v \in \mathbb{V}^N$ and vector $x \in \mathbb{R}^N$, we now denote by $v + x \in \mathbb{V}^N$ the cooperative game defined by $(v + x)(S) = v(S) + x(S) = v(S) + \sum_{i \in S} x_i$ for every coalition $S \in 2^N$. If for some cooperative game $v \in \mathbb{V}^N$, we let $\underline{v}(v) = (v_1, \dots v_n) \in \mathbb{R}^N$, then clearly $v - \underline{v}(v) \in \mathbb{V}_0^N$, which denotes the *zero-normalisation* of the game v.

2.1 Game properties and values

We explore well-known properties of games that are required in our analysis. We refer to a game $v \in \mathbb{V}^N$ as (i) *monotonic* if $v(S) \leqslant v(T)$ for all $S, T \subseteq N$ with $S \subseteq T$ and (ii) *superadditive* if for all $S, T \subseteq N$ with $S \cap T = \varnothing$ it holds that $v(S \cup T) \geqslant v(S) + v(T)$.

The *marginal contribution*—also known as the "utopia" value (Tijs, 1981; Branzei et al., 2008)—of an individual player $i \in N$ in the game $v \in \mathbb{V}^N$ is defined by their marginal or "separable" contribution (Moulin, 1985) to the grand coalition in this game, i.e.,

$$M_i(v) = v(N) - v(N - i). \tag{2}$$

We call a cooperative game $v \in \mathbb{V}^N$ *essential* if it holds that

$$\sum_{j \in N} v_j \leqslant v(N) \leqslant \sum_{j \in N} M_j(v) \tag{3}$$

The class of essential games is denoted as $\mathbb{V}_E^N \subset \mathbb{V}^N$. The class of essential game $\mathbb{V}_E^N$ is also a linear subspace of $\mathbb{V}^N$.[3]

A game $v \in \mathbb{V}^N$ is called *convex* if for all coalitions $S, T \in 2^N$ it holds that $v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T)$ (Shapley, 1971). Driessen (1985) pointed out that a game $v \in \mathbb{V}^N$ is convex if and only if for all $i \in S \subset T$: $v(S) - v(S - i) \leqslant v(T) - v(T - i)$. We denote the subclass of convex games by $\mathbb{V}_C^N \subset \mathbb{V}^N$.

An *allocation*—also known as a "pre-imputation"—in the game $v \in \mathbb{V}^N$ is any point $x \in \mathbb{R}^N$ such that $x(N) = v(N)$. We denote the class of all allocations for the game $v \in \mathbb{V}^N$ by $\mathcal{A}(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\} \neq \varnothing$. We emphasise that allocations can assign positive as well as negative payoffs to individual players in a game. Furthermore, the set of allocations $\mathcal{A}(v) \subset \mathbb{R}^N$ is a linear subspace or manifold in the Euclidean vector space $\mathbb{R}^N$.

[3] We remark that in the literature, the term "weakly essential" or "essential" frequently pertains solely to the first inequality in (3).

An *imputation* in the game $v \in \mathbb{V}^N$ is an allocation $x \in \mathcal{A}(v)$ that is individually rational in the sense that $x_i \geqslant v_i$ for every player $i \in N$. The corresponding imputation set of $v \in \mathbb{V}^N$ is given by $\mathcal{I}(v) = \{x \in \mathcal{A}(v) \mid x_i \geqslant v_i \text{ for all } i \in N\}$. We remark that $\mathcal{I}(v)$ is a polytope in $\mathcal{A}(v)$ for any cooperative game $v \in \mathbb{V}^N$. Furthermore, $\mathcal{I}(v) \neq \varnothing$ if and only if $v(N) \geqslant \sum_{i \in N} v_i$. In particular, this is the case for essential games.

Definition 2.1 *Let $\mathbb{V} \subseteq \mathbb{V}^N$ be some subclass of cooperative games on player set N. A **value** on $\mathbb{V}$ is a function $f \colon \mathbb{V} \to \mathbb{R}^N$ such that $f(v) \in \mathcal{A}(v)$ for every $v \in \mathbb{V}$.*

Note that, since every allocation in a game $v \in \mathbb{V}^N$ distributes the total worth $v(N)$ that is created in that game, a value is defined to be *efficient* in the sense that $\sum_{i \in N} f_i(v) = v(N)$ for all $v \in \mathbb{V}$.

We remark that a value assigns an allocation to every game in the selected subclass, but not necessarily an imputation. We call a value $f \colon \mathbb{V} \to \mathbb{R}^N$ *individually rational* if for every $v \in \mathbb{V} \colon f(v) \in \mathcal{I}(v)$. Hence, an individually rational value on a subclass of games explicitly assigns an imputation to every game in that subclass. We remark that many of the compromise values considered in this paper are actually individually rational.

Finally, for some class of games $\mathbb{V} \subseteq \mathbb{V}^N$, a value $f \colon \mathbb{V} \to \mathbb{R}^N$ is *translation covariant* if $f(v + x) = f(v) + x$ for any game $v \in \mathbb{V}$ and vector $x \in \mathbb{R}^N$.[4]

2.2 The τ-value

Tijs (1981, 1987) seminally introduced the quintessential compromise value, the τ-value. This value is explicitly constructed as an allocation that is the balance of a natural upper bound and a constructed lower bound. In particular, the considered upper bound is the marginal contribution M_i of an individual player $i \in N$ defined by (2). The marginal contribution vector $M(v)$ forms a natural upper bound on the payoff to any player in many cooperative games v.

Using the marginal contribution vector $M(v)$ as an upper bound, Tijs (1981) constructed a natural, corresponding lower bound given by

$$m_i(v) = \max_{S \subseteq N \colon i \in S} R_i(S, v) \qquad \text{where } R_i(S, v) = v(S) - \sum_{j \in S - i} M_j(v) \tag{4}$$

Tijs (1987) refers to $m(v)$ as the "minimal rights" vector for the game v.

Semi-balanced games Following Tijs (1981), we call a game $v \in \mathbb{V}^N$ *semi-balanced* if for every coalition $S \subseteq N$ it holds that

$$v(S) + \sum_{j \in S} v(N - j) \leqslant |S| \, v(N) \tag{5}$$

The class of semi-balanced games is denoted by $\mathbb{V}^N_S \subset \mathbb{V}^N$.

The upper bound M and the lower bound m defined above form a bound pair (m, M) in the sense that for every semi-balanced game $v \in \mathbb{V}^N_S \colon m(v) \leqslant M(v)$, $m(v - m(v)) = 0$ and $M(v - m(v)) = M(v) - m(v)$.

[4] This property is referred to as S-equivalence by Tijs (1981).

Tijs (1981) already showed that there is a close relationship between semi-balanced games and the introduced bound pair (m, M). Indeed, he showed that

$$\mathbb{V}_S^N = \left\{ v \in \mathbb{V}^N \;\middle|\; \sum_{i \in N} m_i(v) \leqslant v(N) \leqslant \sum_{i \in N} M_i(v) \right\}$$

Tijs defined the τ-value as the value on the class of semi-balanced games $\mathbb{V}_S^N$ as the allocation that balances m and M. Hence, the τ-value of a semi-balanced game $v \in \mathbb{V}_S^N$ is now given by $\tau(v) = \lambda_v m(v) + (1 - \lambda_v) M(v)$, where $\lambda_v \in [0, 1]$ is determined such that $\sum_{i \in N} \tau_i(v) = v(N)$. It can be computed that for every $v \in \mathbb{V}_S^N$ and $i \in N$:

$$\tau_i(v) = m_i(v) + \frac{M_i(v) - m_i(v)}{\sum_{j \in N} \big(M_j(v) - m_j(v) \big)} \left(v(N) - \sum_{j \in N} m_j(v) \right) \tag{6}$$

In this paper we set out to develop a generalisation of the notion introduced by Tijs (1981) to general compromise values based on a large class of bound pairs. This is discussed in the next section.

An interesting and successful application of the τ-value is that to two-sided assignment games. Núñez and Rafels (2002) show that the τ-value corresponds exactly to the fair division point—defined as the average of the buyer and seller optimal Core payoff vectors.

3 Compromise Values

We aim to generalise the construction method on which the τ-value is founded to arbitrary bound pairs on well-defined subclasses of cooperative games. In particular, we aim to find bound pairs for which this method results in meaningful compromise values. The determinants of this construction are the chosen lower and upper bounds, which are determined by well-chosen functionals on the relevant class of cooperative games.

3.1 Bound pairs

The next definition incorporates such a construction over the broadest possible collection of classes of cooperative games for which the resulting compromise value is feasible.

Definition 3.1 *Let $\mathbb{V} \subseteq \mathbb{V}^N$ be some class of games on player set N. A pair of functions $(\mu, \eta) \colon \mathbb{V} \to \mathbb{R}^N \times \mathbb{R}^N$ is a **bound pair** on $\mathbb{V}$ if these functions satisfy the following properties:*

(i) *For every $v \in \mathbb{V}$: $\mu(v) \leqslant \eta(v)$, and*

(ii) *For every $v \in \mathbb{V}$: $v - \mu(v) \in \mathbb{V}$ and the following two properties hold:*

  (a) *For every $v \in \mathbb{V}$: $\mu(v - \mu(v)) = 0$, and*

  (b) *For every $v \in \mathbb{V}$: $\eta(v - \mu(v)) = \eta(v) - \mu(v)$.*

*A bound pair (μ, η) is **proper** on $\mathbb{V}$ if there exists at least one game $v \in \mathbb{V}$ with $\mu(v) < \eta(v)$ and it is **strict** on $\mathbb{V}$ if there exists at least one game $v \in \mathbb{V}$ with $\mu(v) \ll \eta(v)$.*

This definition of a bound pair introduces a pair of bound functionals on the vector space of cooperative games. The function μ can be interpreted as a lower bound for every game in the subclass $\mathbb{V} \subseteq \mathbb{V}^N$, while the function η assigns an upper bound. The three properties introduced in the definition of a bound pair ensure that these functions behave as expected from representing lower and upper bounds.

The first property (i) imposes the natural order that an upper bound is at least as large as the lower bound. Properties (ii-a) and (ii-b) are versions of covariance. These properties restrict the choice of bound pairs considerably.

The intuition of property (ii-a) is based on the interpretation of μ as a proper lower bound. For any $v \in \mathbb{V}$, the derived game $v - \mu(v) \in \mathbb{V}^N$ is defined by $(v - \mu(v))(S) = v(S) - \sum_{j \in S} \mu_j(v)$ for coalition $S \subseteq N$. Hence, $v - \mu(v)$ assigns to a coalition the originally generated worth minus the assigned worth of the lower bound to all members of that coalition. It is natural to require that the natural lower bound of $0 \in \mathbb{R}^N$ is assigned to this reduced game. It excludes the assignment of a constant non-zero allocation as a lower bound.

Example 3.2 Consider any $m^0 \in \mathbb{R}^N$ and let $m^0 \colon \mathbb{V}^N \to \mathbb{R}^N$ be the corresponding constant function given by $m^0(v) = m^0$ for every $v \in \mathbb{V}^N$. We remark that the function m^0 does not define a properly configured lower bound as defined above unless $m^0 = 0$. Indeed, $m(v - m(v)) = m^0 \neq 0$ if and only if $m^0 \neq 0$. Hence, we conclude that a constant lower bound violates property (ii-a) in the definition above, unless it is the zero lower bound.
On the other hand, the zero lower bound μ^0 defined by $\mu^0(v) = 0$ for all $v \in \mathbb{V}^N$ is a natural lower bound that can be part of a proper bound pair such as is the case for the PANSC value discussed in Section 3.3.1. ♦

Property (ii-b) in the definition of a bound pair links the upper bound to the lower bound in its functionality. In particular, the upper bound of the derived game $v - \mu(v) \in \mathbb{V}$ is simply the assigned upper bound to the original game v minus the allocated lower bound payoffs. We remark that this property can be interpreted as a specific form of (translation) covariance.

Example 3.3 With reference to Example 3.2, we note that for the zero lower bound μ^0 any upper bound $\eta \colon \mathbb{V}^N \to \mathbb{R}^N$ with $\eta(v) \geqslant 0$ for all $v \in \mathbb{V}^N$ forms a proper bound pair on the whole game space $\mathbb{V}^N$ as defined in Definition 3.1. This allows non-linear upper bounds to be combined with this particular zero lower bound. ♦

On the other hand, some rather natural pairs of bounds might not form proper bound pairs.

Example 3.4 Consider the lower bound $\underline{v}$ that assigns to every game $v \in \mathbb{V}^N$ the vector of individual worths $\underline{v}(v) = (v_1, \ldots, v_n)$.
Considering the upper bound M with $M(v) = (M_1(v), \ldots, M_n(v))$ that assigns the vector of marginal contributions to v, $(\underline{v}, M)$ is a proper bound pair on the class of essential games $\mathbb{V}^N_E$ satisfying all properties of Definition 3.1.
On the other hand, the natural upper bound $\eta^0(v) = (v(N), \ldots, v(N))$ does not form a proper bound pair with $\underline{v}$ on many meaningful classes of games. Although properties (i) and (ii-a) might be

satisfied for a large class of games, $(\underline{v}, \eta^0)$ fails property (ii-b) for any non zero-normalised game $v \in \mathbb{V}^N$, since $\underline{v}(v) \neq 0$,

$$\eta^0(v - \underline{v}(v)\,) = \left(v(N) - \sum_{i \in N} v_i, \ldots, v(N) - \sum_{i \in N} v_i \right)$$
$$\neq (v(N) - v_1, \ldots, v(N) - v_n) = \eta^0(v) - \underline{v}(v).$$

This example shows that even natural lower and upper bounds cannot necessarily always be combined properly into bound pairs. ♦

Linear bound pairs All bound pairs considered in this paper are actually linear in the sense that they are linear functionals on the space of all games $\mathbb{V}^N$. This simply means that these bounds can be written as weighted sums of coalitional worths assigned in the particular game $v \in \mathbb{V}^N$ under consideration.

Formally, a function $f \colon \mathbb{V}^N \longrightarrow \mathbb{R}$ is *linear* if for every game $v \in \mathbb{V}^N$ and every player $i \in N \colon f_i(v) = \alpha^i \cdot v$ for some $\alpha^i \in \mathbb{V}^N$ and the operator "$\cdot$" refers to the inner product on $\mathbb{V}^N$ as a Euclidean vector space. Hence, we can also write $f_i(v) = \sum_{S \subseteq N} \alpha^i_S \, v(S)$ for every $i \in N$ and $v \in \mathbb{V}^N$, using the notation $\alpha^i_S = \alpha^i(S)$ for every $S \subseteq N$.

Definition 3.5 *A pair of functions $(\mu, \eta) \colon \mathbb{V}^N \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$ is a **linear bound pair** if for every $i \in N$ there exist $\mu^i, \eta^i \in \mathbb{V}^N$ with $\mu_i(v) = \mu^i \cdot v$ and $\eta_i(v) = \eta^i \cdot v$ such that*

$$\mu^i = \sum_{j \in N} \left[\sum_{T \colon j \in T} \mu^i_T \right] \mu^j = \sum_{j \in N} \left[\sum_{T \colon j \in T} \eta^i_T \right] \mu^j \tag{7}$$

The next proposition shows that the naming of a pair of linear functions as a "bound pair" in the definition above is justified since for every linear bound pair there is a nonempty linear subspace of V^N such that this pair is indeed a bound pair.

Proposition 3.6 *Every linear bound pair (μ, η) is a bound pair on the non-empty linear subspace $\mathbb{V}(\mu, \eta) = \{v \in \mathbb{V}^N \mid \mu(v) \leqslant \eta(v)\} \neq \varnothing$ of $\mathbb{V}^N$.*

Proof. Let (μ, η) be a linear bound pair satisfying the properties as given.
First, we note that linearity of the two functions μ and η implies that for every $v, w \in \mathbb{V}(\mu, \eta)$ and $\lambda \in \mathbb{R}$ it holds that for every player $i \in N \colon \mu_i(\lambda v + w) = \mu^i \cdot (\lambda v + w) = \lambda \mu^i \cdot v + \mu^i \cdot w = \lambda \mu_i(v) + \mu_i(w) \leqslant \lambda \eta_i(v) + \eta_i(w) = \eta_i(\lambda v + w)$, which shows the linearity of $\mathbb{V}(\mu, \eta)$ as a subspace of $\mathbb{V}^N$. Non-emptiness follows from $v_0 = 0 \in \mathbb{V}(\mu, \eta)$. This shows that (μ, η) indeed satisfies Definition 3.1(i) on $\mathbb{V}(\mu, \eta)$.

Second, we show that (μ, η) satisfies Definition 3.1(ii-a) on $\mathbb{V}^N$. Due to the linearity of μ we only have to show the desired property for every standard base game $b_S \in \mathbb{V}^N$, $S \in 2^N$.

Let $S \subseteq N$. Then we derive for every player $i \in N$ that

$$\mu_i(b_S - \mu(b_S)) = \mu^i \cdot b_S - \mu^i \cdot \mu(b_S) = \mu^i_S - \sum_{T \subseteq N} \mu^i_T \left(\sum_{j \in T} \mu^j_S \right)$$
$$= \mu^i_S - \sum_{j \in N} \left[\sum_{T \subseteq N \colon j \in T} \mu^i_T \right] \mu^j_S = \mu^i_S - \mu^i_S = 0$$

using (7).

Third, to show that (μ, η) satisfies Definition 3.1(ii-b) on $\mathbb{V}^N$, we again can restrict ourselves to checking this property for all base games. Take $b_S \in \mathbb{V}^N$ for some $S \in 2^N$. Now, for every player $i \in N$:

$$\eta_i(b_S - \mu(b_S)) = \eta^i \cdot b_S - \eta^i \cdot \mu(b_S) = \eta^i_S - \sum_{T \subseteq N} \eta^i_T \left(\sum_{j \in T} \mu^j_S \right)$$
$$= \eta^i_S - \sum_{j \in N} \left[\sum_{T \subseteq N \colon j \in T} \eta^i_T \right] \mu^j_S = \eta^i_S - \mu^i_S = \eta_i(b_S) - \mu_i(b_S)$$

using (7). This shows property (ii-b) of Definition 3.1 on $\mathbb{V}^N$. ■

Bound balanced games The following proposition clarifies further when bound pairs are meaningful. Consider $(\mu, \eta) \colon \mathbb{V} \to \mathbb{R}^N \times \mathbb{R}^N$ to be a bound pair on some class of games $\mathbb{V} \subseteq \mathbb{V}^N$ and let $v \in \mathbb{V}$. Then we define

$$Q(v; \mu, \eta) = \{ x \in \mathcal{A}(v) \mid \mu(v) \leqslant x \leqslant \eta(v) \} \tag{8}$$

as the class of allocations of v that are bound by $\mu(v)$ and $\eta(v)$.

Proposition 3.7 *Let $(\mu, \eta) \colon \mathbb{V} \to \mathbb{R}^N \times \mathbb{R}^N$ be a bound pair on $\mathbb{V} \subseteq \mathbb{V}^N$ and let $v \in \mathbb{V}$. For the game v the set of $(\mu(v), \eta(v))$-bound allocations is non-empty if and only if the sums of the lower bound- and upper bound payoffs are lower- and upper bounds for $v(N)$, i.e.,*

$$Q(v; \mu, \eta) \neq \varnothing \quad \text{if and only if} \quad \sum_{i \in N} \mu_i(v) \leqslant v(N) \leqslant \sum_{i \in N} \eta_i(v). \tag{9}$$

Proof. Let (μ, η) be a bound pair on $\mathbb{V} \subseteq \mathbb{V}^N$ and let $v \in \mathbb{V}$.

First, assume that $Q(v; \mu, \eta) \neq \varnothing$. Let $x \in Q(v; \mu, \eta)$. Then $\mu(v) \leqslant x \leqslant \eta(v)$. Moreover, $x \in A(v)$ implies that $x(N) = v(N)$. Therefore, $\sum_{i \in N} \mu_i(v) \leqslant x(N) = v(N) \leqslant \sum_{i \in N} \eta_i(v)$.

Second, assume that $\sum_{i \in N} \mu_i(v) \leqslant v(N) \leqslant \sum_{i \in N} \eta_i(v)$. By Definition 3.1(i), $\mu(v) \leqslant \eta(v)$.
Then it is obvious that there exists some $x \in \mathbb{R}^N$ such that (i) $\sum_{i \in N} x_i = v(N)$ and (ii) $\mu(v) \leqslant x \leqslant \eta(v)$. Clearly, $x \in Q(v; \mu, \eta)$ and, thus, $Q(v; \mu, \eta) \neq \varnothing$.

Together, this shows (9). ■

Proposition 3.7 gives rise to the introduction of the subclass of cooperative games for which the set of bound allocations is non-empty for a given bound pair.

Definition 3.8 *Let (μ, η) be a bound pair on $\mathbb{V} \subseteq \mathbb{V}^N$. The subclass of (μ, η)-**balanced games** is defined by*

$$\mathbb{B}(\mu, \eta) = \left\{ v \in \mathbb{V} \;\middle|\; \sum_{i \in N} \mu_i(v) \leqslant v(N) \leqslant \sum_{i \in N} \eta_i(v) \right\} \tag{10}$$

Equivalently, by Proposition 3.7, the class of (μ, η)-balanced games consists of those games $v \in \mathbb{V}$ for which $Q(v; \mu, \eta) \neq \emptyset$.

The introduction of the subspace of (μ, η)-balanced games gives rise to the question whether compromise points define a value on this class that assigns the corresponding compromise point to each game.

Proposition 3.9 *Let (μ, η) be a bound pair on $\mathbb{V} \subseteq \mathbb{V}^N$ such that the corresponding subclass of (μ, η)-balanced games $\mathbb{B}(\mu, \eta) \subseteq \mathbb{V}$ is non-empty. Then for every (μ, η)-balanced game $v \in \mathbb{B}(\mu, \eta)$:*
If $\mu(v) < \eta(v)$, it holds that

$$\gamma(v; \mu, \eta) = \frac{v(N) - \sum_{i \in N} \mu_i(v)}{\sum_{i \in N} (\eta_i(v) - \mu_i(v))} \, \eta(v) + \frac{\sum_{i \in N} \eta_i(v) - v(N)}{\sum_{i \in N} (\eta_i(v) - \mu_i(v))} \, \mu(v) \in Q(v; \mu(v), \eta(v)) \tag{11}$$

and, if $\mu(v) = \eta(v)$, it holds that

$$\gamma(v; \mu, \eta) = \mu(v) = \eta(v) \in Q(v; \mu(v), \eta(v)) \tag{12}$$

The map $\gamma(\cdot; \mu, \eta) \colon \mathbb{B}(\mu, \eta) \to \mathbb{R}^N$ defines a value on $\mathbb{B}(\mu, \eta)$, satisfying $\sum_{i \in N} \gamma_i(v; \mu, \eta) = v(N)$ for every $v \in \mathbb{B}(\mu, \eta)$.

The proof of Proposition 3.9 is trivial and is, therefore, omitted.[5] We refer to the value $\gamma(\cdot; \mu, \eta)$ on $\mathbb{B}(\mu, \eta)$ introduced in Proposition 3.9 as the (μ, η)-*compromise value.*

3.2 An axiomatic characterisation of compromise values

We continue our discussion of compromise values by constructing an axiomatic characterisation of *all* compromise values based on bound pairs as introduced in Definition 3.1.

In particular, we are able to show that the axiomatisation of the τ-value seminally developed by Tijs (1981) can be extended to any arbitrary compromise value. The next theorem provides a complete characterisation of compromise values in terms of the associated bound pair.

Theorem 3.10 *Let (μ, η) be a bound pair on $\mathbb{V} \subseteq \mathbb{V}^N$ and let $\mathbb{B}(\mu, \eta) \subseteq \mathbb{V}$ be the corresponding subclass of (μ, η)-balanced games.*
Then the (μ, η)-compromise value $\gamma(\cdot; \mu, \eta)$ is the unique value $f \colon \mathbb{B}(\mu, \eta) \to \mathbb{R}^N$ which satisfies the following two properties:

[5] The proof can be obtained from the authors on request.

(i) ***Minimal rights property:***
For every $v \in \mathbb{B}(\mu,\eta)$: $f(v) = f(v - \mu(v)) + \mu(v)$.

(ii) ***Restricted proportionality:***
For every game $v \in \mathbb{B}(\mu,\eta)$ *with* $\mu(v) = 0$ *there exists some* $\lambda_v \in \mathbb{R}$ *such that* $f(v) = \lambda_v\, \eta(v)$.

Proof. Let (μ,η) be a bound pair on $\mathbb{V} \subseteq \mathbb{V}^N$ and let $\mathbb{B}(\mu,\eta) \subseteq \mathbb{V}$ be the corresponding subclass of (μ,η)-balanced games.

We show that $\gamma(\cdot;\mu,\eta)$ satisfies the properties stated in the assertion. First, note that $\gamma(\cdot;\mu,\eta)$ is a value, since it is efficient, i.e., $\sum_{i\in N} \gamma_i(v;\mu,\eta) = v(N)$ for every $v \in \mathbb{B}(\mu,\eta)$.
Second, with reference to Proposition 3.9, $\gamma(\cdot;\mu,\eta)$ satisfies restricted proportionality on $\mathbb{B}(\mu,\eta) \subseteq \mathbb{V}^N$. Indeed, if $\mu(v) = 0$ for $v \in \mathbb{B}(\mu,\eta)$, it follows that

$$\gamma(v;\mu,\eta) = \frac{v(N)}{\sum_{i\in N} \eta_i(v)}\, \eta(v)$$

Third, to show the minimal rights property, let $v \in \mathbb{B}(\mu,\eta)$. Then by Definition 3.1, $\mu(v - \mu(v)) = 0$ and $\eta(v - \mu(v)) = \eta(v) - \mu(v)$. We distinguish two cases:
If $\mu(v) = \eta(v)$, then $\eta(v - \mu(v)) = 0 = \mu(v - \mu(v))$. Hence, by definition of γ, for some $\lambda_1 \in [0,1]$:

$$\gamma(v - \mu(v);\mu,\eta) = \lambda_1 \eta(v - \mu(v)) + (1 - \lambda_1)\mu(v - \mu(v)) = 0$$

Therefore, $\gamma(v;\mu,\eta) = \mu(v) = \gamma(v - \mu(v);\mu,\eta) + \mu(v)$.
If $\mu(v) < \eta(v)$, we have from $\mu(v - \mu(v)) = 0$ and $\eta(v - \mu(v)) = \eta(v) - \mu(v)$ by (11) that

$$\begin{aligned}\gamma(v - \mu(v);\mu,\eta) &= \frac{v(N) - \sum_{i\in N} \mu_i(v) - \sum_{i\in N} \mu_i(v - \mu(v))}{\sum_{i\in N} \left(\eta_i(v - \mu(v)) - \mu_i(v - \mu(v))\right)}\eta(v - \mu(v)) \\ &= \frac{v(N) - \sum_{i\in N} \mu_i(v)}{\sum_{i\in N} \left(\eta_i(v) - \mu_i(v)\right)}\, [\eta(v) - \mu(v)\,]\end{aligned}$$

Hence,

$$\begin{aligned}\gamma(v - \mu(v);\mu,\eta) + \mu(v) &= \frac{v(N) - \sum_{i\in N} \mu_i(v)}{\sum_{i\in N} \left(\eta_i(v) - \mu_i(v)\right)}\, \eta(v) + \left[1 - \frac{v(N) - \sum_{i\in N} \mu_i(v)}{\sum_{i\in N} \left(\eta_i(v) - \mu_i(v)\right)}\right] \mu(v) \\ &= \frac{v(N) - \sum_{i\in N} \mu_i(v)}{\sum_{i\in N} \left(\eta_i(v) - \mu_i(v)\right)}\, \eta(v) - \frac{v(N) - \sum_{i\in N} \eta_i(v)}{\sum_{i\in N} \left(\eta_i(v) - \mu_i(v)\right)}\, \mu(v) \\ &= \gamma(v;\mu,\eta).\end{aligned}$$

This shows that the compromise value γ satisfies the minimal rights property.

Next, we show that if a value $f\colon \mathbb{B}(\mu,\eta) \to \mathbb{R}^N$ satisfies the two stated properties, it necessarily has to be the corresponding compromise value.
Combining the two properties for any $v \in \mathbb{B}(\mu,\eta)$ we derive that

$$f(v) = f(v - \mu(v)) + \mu(v) = \lambda\, \eta(v - \mu(v)) + \mu(v)$$

for some $\lambda \in \mathbb{R}$ where the first equality follows from the minimal rights property and the second equality from restricted proportionality. Using $\eta(v - \mu(v)) = \eta(v) - \mu(v)$ we subsequently conclude that

$$f(v) = \lambda\,\eta(v) + (1 - \lambda)\mu(v).$$

If $\mu(v) = \eta(v)$, it follows immediately that $f(v) = \mu(v) = \gamma(v; \mu, \eta)$.
This leaves the case that $\mu(v) < \eta(v)$. Using the efficiency of $f(v)$, it holds that $\sum_{i \in N} f_i(v) = v(N)$. Together with $\sum_{i \in N} \mu_i(v) \leqslant v(N) = \sum_{i \in N} f_i(v) \leqslant \sum_{i \in N} \eta_i(v)$ and $\mu(v) < \eta(v)$ it follows that

$$\sum_{i \in N} f_i(v) = \lambda \sum_{i \in N} \eta_i(v) + (1 - \lambda) \sum_{i \in N} \mu_i(v) = v(N)$$

implying that

$$\lambda \left(\sum_{i \in N} (\eta_i(v) - \mu_i(v)) \right) = v(N) - \sum_{i \in N} \mu_i(v)$$

Hence,

$$\lambda = \frac{v(N) - \sum_{i \in N} \mu_i(v)}{\sum_{i \in N} (\eta_i(v) - \mu_i(v))}$$

showing that $f(v) = \gamma(v; \mu, \eta)$. ■

3.3 Three illustrations of compromise values

To elucidate and refine the concepts explored in the preceding discussion, we examine three compromise values. Notably, the first two values, PANSC and Gately values, have been previously examined in the literature. In contrast, the KM-value represents a novel contribution, emerging from the integration of a plausible upper and lower bound, applicable over the entire space of all games $\mathbb{V}^N$.

3.3.1 The PANSC value

Considering the zero vector as the chosen lower bound and the marginal contributions vector as the chosen upper bound, the resulting compromise value is the *Proportional Allocation of Non-Separable Contributions* (PANSC) value. This compromise value has been studied extensively by van den Brink et al. (2023).

The PANSC value assigns to every player a payoff that is proportional to the player's marginal contribution to the total wealth generated in the game. Formally, for the game $v \in \mathbb{V}^N$ with $\sum_{j \in N} M_j(v) \neq 0$ and player $i \in N$, the PANSC value is defined by

$$\text{PANSC}_i(v) = \frac{M_i(v)}{\sum_{j \in N} M_j(v)}\, v(N). \tag{13}$$

We claim that the PANSC value corresponds actually to the (μ^0, M)-compromise value, where

$\mu^0(v) = 0$ and $M(v) = (M_1(v), \dots, M_n(v)\,)$ for every $v \in \mathbb{B}(\mu^0, M)$ with

$$\mathbb{B}(\mu^0, M) = \left\{ v \in \mathbb{V}^N \;\middle|\; 0 \leqslant v(N) \leqslant \sum_{j \in N} M_j(v) \right\} \tag{14}$$

We remark that this class of cooperative games includes the set of non-negative essential games.

It is easy to show that PANSC: $\mathbb{B}(\mu^0, M) \to \mathbb{R}^N$ is the unique value f on $\mathbb{B}(\mu^0, M)$ such that for every game $v \in \mathbb{B}(\mu^0, M)$ there exists some $\lambda_v \geqslant 0$ with $f(v) = \lambda_v M(v)$.

3.3.2 The Gately value

Another example of a compromise value is the *Gately value* initially proposed by Gately (1974) and further developed by Littlechild and Vaidya (1976); Charnes et al. (1978); Staudacher and Anwander (2019) and Gilles and Mallozzi (2024).

The Gately value can be understood as the compromise value based on the bound pair $(\underline{v}, M)$, where, for every $v \in \mathbb{V}^N$, $\underline{v}(v) = (v_1, \dots, v_n)$ is the vector of individual payoffs and $M(v) = (M_1(v), \dots, M_n(v))$ is the vector of marginal contributions. The pair $(\underline{v}, M)$ indeed forms a bound pair[6] on the generated class of $(\underline{v}, M)$-balanced games $\mathbb{B}(\underline{v}, M)$, which is exactly the class of essential games $\mathbb{V}^N_E$.

The corresponding $(\underline{v}, M)$-compromise value on $\mathbb{B}(\underline{v}, M) = \mathbb{V}^N_E$ is the Gately value given by

$$g_i(v) = v_i + \frac{M_i(v) - v_i}{\sum_{j \in N} \left(M_j(v) - v_j \right)} \left(v(N) - \sum_{j \in N} v_j \right) \tag{15}$$

for every essential game $v \in \mathbb{V}^N_E$ and player $i \in N$.[7]

The Gately value has some interesting properties. First, the Gately value is the value at which the so-called propensities to disrupt for all players are minimal (Gately, 1974). This refers to the interpretation that the Gately value is the equilibrium outcome of a bargaining process over the allocation of the generated worths in the cooperative game.

Second, the Gately value is *self-dual* in the sense that the Gately value of the dual game is equal to the Gately value of the original game (Gilles and Mallozzi, 2024, Proposition 3.9).

Third, the Gately value is in the Core of every three player cooperative game as shown by (Gilles and Mallozzi, 2024, Theorem 4.2). For games with more than three players, this might not be the case, showing that in general the relationship between compromise values and the Core is undetermined.

3.3.3 The KM-value

It is an interesting question whether a compromise value can be constructed on the whole class of cooperative games $\mathbb{V}^N$. This has been investigated by van den Brink (1994), who constructed a

[6] It is relatively easy to check the three required properties of a bound pair for $(\underline{v}, M)$.

[7] With reference to the discussion in Sections 4 and 5 in this paper, we remark that the Gately value cannot be constructed through the methods discussed there. Hence, the Gately value does not result from either the lower bound $\underline{v}$ or the upper bound M.

value based on a lower bound introduced by Kikuta (1980) and an upper bound that was already considered by Milnor (1952). Both of these bounds were originally considered for the Core only.

Formally, the Kikuta lower bound $\underline{M}\colon \mathbb{V}^N \to \mathbb{R}^N$ assigns to every player $i \in N$ her minimal marginal contribution in a game $v \in \mathbb{V}^N$, defined as

$$\underline{M}_i(v) = \min_{S \subseteq N\colon i \in S} (\, v(S) - v(S-i) \,) \tag{16}$$

Similarly, the Milnor upper bound $\overline{M}\colon \mathbb{V}^N \to \mathbb{R}^N$ assigns to every player $i \in N$ his maximal marginal contribution in a game $v \in \mathbb{V}^N$, defined as

$$\overline{M}_i(v) = \max_{S \subseteq N\colon i \in S} (\, v(S) - v(S-i) \,) \tag{17}$$

The next claim summarises the properties of these bounds and introduces the *KM-value* as the $(\underline{M}, \overline{M})$-compromise value.

Proposition 3.11 *The pair $(\underline{M}, \overline{M})$ forms a bound pair on $\mathbb{B}(\underline{M}, \overline{M}) = \mathbb{V}^N$.*

Proof. To show that $(\underline{M}, \overline{M})$ is a bound pair on $\mathbb{V}^N$, we show the conditions (i), (ii) and (iii) of Definition 3.1 over the whole game space $\mathbb{V}^N$.
(i) It is immediately clear that for every $v \in \mathbb{V}^N\colon \underline{M}(v) \leqslant \overline{M}(v)$.
(ii) Next, let $v \in \mathbb{V}^N$ and let $v' = v - \underline{M}(v)$. To check that $\underline{M}(v - \underline{M}(v)) = \underline{M}(v') = 0$ for $i \in N$, let $S_i \in 2^N$ be such that $i \in S_i$ and $\underline{M}_i(v) = v(S_i) - v(S_i - i)$. Then for an arbitrary coalition $S \in 2^N$ we derive that

$$v'(S) = (v - \underline{M}(v))(S) = v(S) - \sum_{j \in S} \underline{M}_j(v) = v(S) - \sum_{j \in S} v(S_j) + \sum_{j \in S} v(S_j - j). \tag{18}$$

Now for any player $i \in S$ it is easy to see that

$$v'(S) - v'(S-i) = v(S) - v(S-i) - \underline{M}_i(v). \tag{19}$$

This, in turn, implies that

$$\begin{aligned}\underline{M}_i(v') &= \min_{S \in 2^N : i \in S} [v'(S) - v'(S-i)] \\ &= \min_{S \in 2^N : i \in S} (v(S) - v(S-i)) - \underline{M}_i(v) \\ &= \underline{M}_i(v) - \underline{M}_i(v) = 0\end{aligned}$$

where the second equality follows by definition and the third equality follows from (19). Thus, (ii) is satisfied.
It is easily established that for every $v \in \mathbb{V}^N\colon \overline{M}(v - \underline{M}(v)) = \overline{M}(v) - \underline{M}(v)$. Hence, with (i) and (ii) above, $(\underline{M}, \overline{M})$ is shown to be a bound pair on $\mathbb{V}^N$.

To establish that $\mathbb{B}(\underline{M}, \overline{M}) = \mathbb{V}^N$ we introduce $N_0 = \varnothing$ and for every $k = 1, \dots, n\colon N_k = \{1, \dots, k\} \subseteq N$. In particular, $N_n = N$. Hence, for every game $v \in \mathbb{V}^N$ and every player $i \in N$ we have that

$\underline{M}_i(v) \leqslant v(N_i) - v(N_{i-1}) \leqslant \overline{M}_i(v)$. This implies that

$$\sum_{i \in N} \underline{M}_i(v) \leqslant \sum_{k \in N} \left(v(N_k) - v(N_{k-1}) \right) = v(N_n) = v(N) \leqslant \sum_{i \in N} \overline{M}_i(v).$$

This shows that indeed $\mathbb{B}(\underline{M}, \overline{M}) = \mathbb{V}^N$. ∎

We refer to the $(\underline{M}, \overline{M})$-compromise value $\kappa \colon \mathbb{V}^N \to \mathbb{R}^N$ as the *KM-value*. It is a compromise value that is defined for *all* cooperative games.

We note that the KM-value κ is also self-dual in the sense that $\kappa(v) = \kappa(v^*)$ for all $v \in \mathbb{V}^N$. Indeed, we note that for $v \in \mathbb{V}^N$ and $i \in N$:

$$\begin{aligned} \underline{M}_i(v^*) &= \min_{S \colon i \in S} \left(v^*(S) - v^*(S-i) \right) = \min_{S \colon i \in S} \left(v(N) - v(N \setminus S) - v(N) + v(N \setminus (S-i)) \right) \\ &= \min_{S \colon i \in S} \left(v((N \setminus S) + i) - v(N \setminus S) \right) = \min_{T \colon i \in T} \left(v(T) - v(T-i) \right) = \underline{M}_i(v). \end{aligned}$$

Hence, $\underline{M}(v^*) = \underline{M}(v)$. Similarly, we can show that $\overline{M}(v^*) = \overline{M}(v)$. This implies that indeed $\kappa(v^*) = \kappa(v)$.

The KM-value for convex games With reference to the discussion of the τ-value in Section 2, we remark that all convex games are semi-balanced, i.e., $\mathbb{V}_C^N \subset \mathbb{V}_S^N$, as pointed out by Tijs (1981). The next proposition addresses the nature of the KM-value and the τ-value for this class of games. In particular, for convex games, the τ-value is equal to the KM-value.

Proposition 3.12 *For every convex game $v \in \mathbb{V}_C^N$: $\kappa(v) = \tau(v)$.*

Proof. Let $v \in \mathbb{V}_C^N$ be a convex game and let $i \in S \subset T$. Then by convexity it follows immediately that for any player $i \in N$:

$$\underline{M}_i(v) = \min_{S \colon i \in S} \left(v(S) - v(S-i) \right) = v_i = m_i(v),$$

where we use the fact that $m_i(v) = v_i$ for any convex game as shown by Driessen and Tijs (1985), and

$$\overline{M}_i(v) = \max_{S \colon i \in S} \left(v(S) - v(S-i) \right) = v(N) - v(N-i) = M_i(v).$$

Therefore, $\kappa(v) = \gamma(v; \underline{M}, \overline{M}) = \gamma(v; m, M) = \tau(v)$. ∎

We have introduced compromise values for arbitrary bound pairs. We can also construct proper bound pairs from either a lower bound function or an upper bound function. This results in certain natural bound pairs that lead to compromise values that have been considered in the literature on cooperative games. This is explored in the next sections.

4 Constructing compromise values from lower bounds

In this section we examine a specific subset of compromise values derived solely from an imposed lower bound. While this construction is not unique, the methodology employed here appears particularly intuitive. We do this by introducing a specific procedure how to associate to every lower bound function an appropriate upper bound function.

4.1 Regular lower bounds and LBC values

The next definition introduces a category of "regular" lower bounds that satisfy condition (ii-a) of Definition 3.1, from which compromise values can be constructed in a relatively straightforward manner.

Definition 4.1 *The function $\mu \colon \mathbb{V}^N \to \mathbb{R}^N$ is referred to as a* ***regular lower bound*** *if for every $v \in \mathbb{B}_\ell(\mu)$ it holds that $\mu(v - \mu(v)) = 0$, where*

$$\mathbb{B}_\ell(\mu) := \left\{ v \in \mathbb{V}^N \;\middle|\; \sum_{i \in N} \mu_i(v) \leqslant v(N) \right\} \tag{20}$$

is the subclass of μ-lower bound games.

With reference to the discussion of linear bound pairs in Section 3.1, we refer to a lower bound μ_i for some player $i \in N$ as *linear* if there exists some $\mu^i \in \mathbb{V}^N$ such that $\mu_i(v) = \mu^i \cdot v$ for all $v \in \mathbb{V}^N$. The next property identifies the defining characteristics of linear regular lower bounds, linking this to the properties imposed on a linear bound pair.

Proposition 4.2 *Let $\mu \colon \mathbb{B}_\ell(\mu) \to \mathbb{R}^N$ be some linear lower bound in the sense that $\mu_i(v) = \mu^i \cdot v$ for all games $v \in \mathbb{B}_\ell(\mu)$ and players $i \in N$ with $\mu^i \in \mathbb{V}^N$. Then μ is a regular lower bound if and only if for every $i \in N$:*

$$\mu^i = \sum_{j \in N} \left[\sum_{T \colon j \in T} \mu^i_T \right] \mu^j \tag{21}$$

Proof. The proof of the assertion proceeds in the same fashion as the first part of the proof of Proposition 3.6. In particular, we need to show that $\mu(v - \mu(v)) = 0$ for every linear lower bound μ. Due to linearity, it suffices to show this for every base game $b_S \in \mathbb{V}^N$ for $S \subseteq N$.
Let $S \subseteq N$ and consider any $i \in N$. Then

$$\begin{aligned} \mu_i(b_S - \mu(b_S)) = \mu^i \cdot b_S - \mu^i \cdot \mu(v) &= \mu^i_S - \sum_{T \subseteq N} \mu^i_T \left(\sum_{j \in T} \mu^j_S \right) \\ &= \mu^i_S - \sum_{j \in N} \left[\sum_{T \colon j \in T} \mu^i_T \right] \mu^j_S = \mu^i_S - \mu^i_S = 0 \end{aligned}$$

if and only if the hypothesis (21) stated in the assertion holds. ∎

The next proposition links the regularity of a lower bound to the ability to construct a natural upper bound with this lower bound such that the resulting pair forms a bound pair. This gives rise to the identification of a natural compromise value for any given regular lower bound.

Proposition 4.3 *Let $\mu\colon \mathbb{V}^N \to \mathbb{R}^N$ be a regular lower bound on $\mathbb{B}_\ell(\mu)$.*

(a) *The function $\eta^\mu\colon \mathbb{B}_\ell(\mu) \to \mathbb{R}^N$ defined by*

$$\eta_i^\mu(v) = v(N) - \sum_{j\neq i} \mu_i(v) \tag{22}$$

forms a bound pair with μ on $\mathbb{B}_\ell(\mu)$ in the sense that μ is the lower bound and η^μ is the upper bound over the corresponding class of (μ, η^μ)-balanced games $\mathbb{B}(\mu, \mu^\eta) = \mathbb{B}_\ell(\mu) \subset \mathbb{V}^N$.

(b) *The corresponding (μ, η^μ)-compromise value on $\mathbb{B}_\ell(\mu)$ is given by*

$$\gamma_i(v;\mu) = \mu_i(v) + \frac{1}{n}\left[v(N) - \sum_{j\in N} \mu_j(v)\right] \tag{23}$$

*The value $\gamma(\cdot;\mu)$ can be denoted as the μ-**Lower Bound Compromise** (μ-LBC) value on $\mathbb{B}_\ell(\mu)$.*

Proof. Let $\mu\colon \mathbb{V}^N \to \mathbb{R}^N$ be a regular lower bound, i.e., $\mu(v - \mu(v)) = 0$.

(a) We show that the upper bound $\eta^\mu\colon \mathbb{B}_\ell(\mu) \to \mathbb{R}^N$ defined by (22) satisfies, in conjunction with μ as the lower bound, the conditions of Definition 3.1:

(i) It is obvious that $\mu(v) \leqslant \eta^\mu(v)$ for $v \in \mathbb{B}_\ell(\mu)$.

(ii-a) Condition 3.1(ii-a) follows from the assumption that μ is a regular lower bound.

(ii-b) To show condition 3.1(ii-b), note that for $v \in \mathbb{B}_\ell(\mu)$:

$$\begin{aligned}\eta_i^\mu(v-\mu(v)) &= (v-\mu(v))(N) - \sum_{j\neq i} \mu_j(v-\mu(v)) \\ &= (v-\mu(v))(N) = v(N) - \sum_{j\in N} \mu(v) = \eta_i^\mu(v) - \mu_i(v).\end{aligned}$$

Therefore, (μ, η^μ), indeed, forms a bound pair as defined in Definition 3.1.

To show that $\mathbb{B}(\mu, \eta^\mu) = \mathbb{B}_\ell(\mu)$, we note that for every $v \in \mathbb{B}_\ell(\mu)$:

$$\begin{aligned}\sum_{i\in N} \eta_i^\mu(v) = \sum_{i\in N}\left[v(N) - \sum_{j\neq i}\mu_j(v)\right] &= n\,v(N) - (n-1)\sum_{j\in N}\mu_j(v) \\ &= v(N) + (n-1)\left[v(N) - \sum_{j\in N}\mu_j(v)\right] \geqslant v(N),\end{aligned}$$

showing that $\mathbb{B}(\mu, \eta^\mu) = \mathbb{B}_\ell(\mu)$ is the corresponding class of (μ, η^μ)-balanced games.

(b) Next, we show that $\gamma(\cdot;\mu)$ defined by (23) is the corresponding (μ, η^μ)-compromise value on

$\mathbb{B}_\ell(\mu)$. For every $v \in \mathbb{B}_\ell(\mu)$:

$$v(N) - \sum_{j\in N} \mu_j(v) = \eta_i^\mu(v) - \mu_i(v) \qquad \text{for any } i \in N, \text{ and}$$

$$\sum_{j\in N} \eta_j^\mu(v) - v(N) = (n-1)\left[v(N) - \sum_{j\in N} \mu_j(v)\right].$$

If $\mu(v) = \eta^\mu(v)$, by definition, $v(N) = \sum_{i\in N} \mu_i(v)$. Hence, the corresponding (μ, η^μ)-compromise value is trivially $\gamma(v;\mu) = \mu(v)$, confirming (23) for this case.
For $\mu(v) < \eta^\mu(v)$, with the above and (11), we now compute that the (μ, η^μ)-compromise value $\gamma(v) = \gamma(v;\mu,\eta)$ on $\mathbb{B}(\mu,\eta^\mu) = \mathbb{B}_\ell(\mu)$ is given by

$$\begin{aligned}
\gamma_i(v) &= \frac{v(N) - \sum_{j\in N}\mu_j(v)}{\sum_{j\in N}(\eta_j^\mu(v) - \mu_j(v))}\,\eta_i^\mu(v) + \frac{\sum_{j\in N}\eta_j^\mu(v) - v(N)}{\sum_{j\in N}(\eta_j^\mu(v) - \mu_j(v))}\,\mu_i(v) \\
&= \frac{v(N) - \sum_{j\in N}\mu_j(v)}{n(v(N) - \sum_{j\in N}\mu_j(v))}\left(v(N) - \sum_{j\neq i}\mu_j(v)\right) + \frac{(n-1)(v(N) - \sum_{j\in N}\mu_j(v))}{n(v(N) - \sum_{j\in N}\mu_j(v))}\mu_i(v) \\
&= \tfrac{1}{n}\left[v(N) - \sum_{j\neq i}\mu_j(v)\right] + \tfrac{n-1}{n}\,\mu_i(v) \\
&= \tfrac{1}{n}\left[v(N) - \sum_{j\in N}\mu_j(v)\right] + \mu_i(v) = \gamma_i(v;\mu)
\end{aligned}$$

for any $i \in N$.
This completes the proof of the proposition. ■

The method introduced in Proposition 4.3 to derive an upper bound to any regular lower bound is intuitive and partly follows the same reasoning as the marginal contribution is taken as upper bound in the definition of the τ-value. The main difference is that the marginal contribution of a player subtracts all individual worths of the other players from $v(N)$. To introduce η^μ as an upper bound, we subtract all individual lower bounds of the other players from $v(N)$

When the lower bound μ and the upper bound η are fixed, (11) yields a specific value for each game $v \in \mathbb{B}(\mu,\eta)$. In such instances, we refer to the value γ that assigns allocations $\gamma(v)$ to games within this class as the (μ,η)-compromise value. More generally, the value depends on the chosen lower bound μ and upper bound η, leading us to denote it as $\gamma(v;\mu,\eta)$ in Equation (11). As demonstrated in Proposition 4.3, for any regular lower bound μ, the upper bound η^μ is derived from μ, rendering the value dependent solely on μ. Consequently, we denote it as $\gamma(v;\mu)$ in this scenario. Nevertheless, it is important to emphasise that our ultimate interest lies in the values $\gamma(v)$.

4.2 A characterisation of lower bound based compromise values

The axiomatisation of arbitrary compromise values—developed in Theorem 3.10—can be sharpened for the subclass of compromise values that can be constructed from a lower bound functional. The next theorem states that the restricted proportionality property of Theorem 3.10 can be replaced by an egalitarian division property.

Theorem 4.4 *Let $\mu\colon \mathbb{V}^N \to \mathbb{R}^N$ be a regular lower bound on the class of μ-bound games $\mathbb{B}_\ell(\mu)$. Then the μ-LBC value is the unique value $f\colon \mathbb{B}_\ell(\mu) \to \mathbb{R}^N$ that satisfies the following two properties:*

(i) ***Minimal rights property:***
For every $v \in \mathbb{B}_\ell(\mu)\colon f(v) = f(v - \mu(v)) + \mu(v)$.

(ii) ***Egalitarian Division property:***
For every game v with $\mu(v) = 0$, there exists some $\lambda_v \in \mathbb{R}$ such that $f(v) = \lambda_v\, e$, where $e = (1, \ldots, 1)$.

Proof. It is obvious that any μ-LBC value indeed satisfies these two listed properties.
Next, let $f\colon \mathbb{B}_\ell(\mu) \to \mathbb{R}^N$ be some function that satisfies the two listed properties. First, for $v \in \mathbb{B}_\ell(\mu)$ with $\mu(v) = 0$, by efficiency of the value f and the egalitarian division property it holds that $f(v) = \frac{v(N)}{n}\, e$.
Next, let $v \in \mathbb{B}_\ell(\mu)$ be arbitrary. Then for every $i \in N$, by the minimal rights property and the fact that $\mu(v - \mu(v)\,) = 0$, it follows that

$$f_i(v) = f_i(v - \mu(v)\,) + \mu_i(v) = \frac{(v - \mu(v)\,)(N)}{n} + \mu_i(v)$$
$$= \frac{1}{n}\left(v(N) - \sum_{j \in N} \mu_j(v) \right) + \mu_i(v) = \gamma_i(v; \mu)$$

This shows the assertion. ∎

4.3 Two well-known LBC values

We discuss two well-known compromise values that are based on the construction method set out in Proposition 4.3, based on a well-defined regular lower bound only. These LBC values are the egalitarian value and the CIS value.

4.3.1 The Egalitarian Value

A trivial lower bound is the function $\mu^0\colon \mathbb{V}^N \to \mathbb{R}^N$ with $\mu^0(v) = 0$ for every $v \in \mathbb{V}^N$. Clearly, any function $\eta\colon \mathbb{V}^N \to \mathbb{R}^N$ with $\eta(v) \geqslant 0$ for $v \in \mathbb{V} \subset \mathbb{V}^N$ is a non-trivial upper bound for μ^0 to form a bound pair, where $\mathbb{V} \subset \mathbb{V}^N$ is a subclass of games v for which $\mathcal{A}(v)$ has a non-empty relative interior.

Indeed, the upper bound constructed in Proposition 4.3(a) is the corresponding upper bound η^0 given by $\eta^0(v) = (\,v(N), \ldots, v(N)\,) \in \mathbb{R}^N$. This corresponds to the trivial upper bound assigning the total wealth generated in the corresponding cooperative game $v \in \mathbb{V}^N$ to every player in the game as an upper bound on their payoff. The class of (μ^0, η^0)-balanced games is now given by

$$\mathbb{B}(\mu^0, \eta^0) = \mathbb{B}_\ell(\mu^0) = \{v \in \mathbb{V}^N \mid v(N) \geqslant 0\}. \tag{24}$$

The corresponding (μ^0, η^0)-compromise value given by Proposition 4.3(b) is the *Egalitarian Value*

$E\colon \mathbb{B}_\ell(\mu^0) \longrightarrow \mathbb{R}^N$ defined by[8]

$$E_i(v) = \frac{v(N)}{n} \qquad \text{for every } i \in N. \tag{25}$$

The egalitarian value E is the unique value on $\mathbb{B}_\ell(\mu^0)$ such that there exists some $\lambda \geqslant 0$ with $E(v) = \lambda\, \eta^0(v)$, i.e., $E_i(v) = \lambda\, v(N)$. Clearly, $\lambda = \frac{1}{n}$.

4.3.2 The CIS-value

Another lower bound that is considered widely in the literature on cooperative games is that of the vector of the individual worths in a game. It forms a natural lower bound on allocated payoffs and many values considered in the literature indeed have this vector as a lower bound on the assigned payoffs.

Formally, for any cooperative game $v \in \mathbb{V}^N$ this lower bound is described by the vector $\underline{\nu}(v) = (v_1, \dots, v_n) \in \mathbb{R}^N$. Since $\underline{\nu}_i(v - \underline{\nu}(v)\,) = v_i - v_i = 0$ for any $i \in N$, the natural lower bound $\underline{\nu}$ is regular. The corresponding class of $\underline{\nu}$-lower bound games is now identified as

$$\mathbb{B}_\ell(\underline{\nu}) = \left\{ v \in \mathbb{V}^N \,\middle|\, \sum_{i \in N} v_i \leqslant v(N) \right\} \tag{26}$$

which includes the class of essential games. We note that $\mathbb{B}_\ell(\underline{\nu})$ is the class of games that admit a non-empty set of imputations.

Next, using Proposition 4.3(a), we can construct the corresponding upper bound $\eta'\colon \mathbb{V}^N \longrightarrow \mathbb{R}^N$ which for every $i \in N$ is defined by

$$\eta'_i(v) = v(N) - \sum_{j \neq i} v_j \tag{27}$$

The resulting $(\underline{\nu}, \eta')$-compromise value as constructed in Proposition 4.3(b) is the *Centre-of-gravity of the Imputation Set* (CIS) considered by Driessen and Funaki (1991), which on the class of $\underline{\nu}$-lower bound games $\mathbb{B}_\ell(\underline{\nu})$ for every $i \in N$ is defined by

$$\mathrm{CIS}_i(v) = v_i + \tfrac{1}{n} \left(v(N) - \sum_{j \in N} v_j \right) \tag{28}$$

It can easily be verified that the CIS-value is indeed equal to the $(\underline{\nu}, \eta')$-compromise value as already remarked by van den Brink (1994).[9]

[8] Axiomatisations of this Egalitarian value using axioms similar as those for the Shapley value, are given in van den Brink (2007).

[9] We refer also to Driessen and Funaki (1991); Dragan et al. (1996); van den Brink and Funaki (2009); Hou et al. (2019) and Zou et al. (2022) for discussions of the CIS-value and related concepts from different perspectives.

5 Constructing compromise values from upper bounds

In Proposition 4.3, we provided a method to derive an associated upper bound using (22). We also introduced a method to construct a compromise value based on a regular lower bound. In this section, we explore the possibility of constructing compromise values from a given upper bound. For a given covariant upper bound η, we construct a proper corresponding lower bound μ^η such that (μ^η, η) forms a proper bound pair.

Our construction method is based on the methodology developed by Tijs (1981) for calculating the τ-value. Bergantiños and Massó (1996) further developed this approach by combining the Milnor upper bound $\overline{M}$ with the Tijs lower bound construction method. Sánchez-Soriano (2000) extended this research and introduced a general construction method based on this methodology. Here, we further generalise this method.

5.1 Covariant upper bounds and UBC values

We note that the methodology as set out by Sánchez-Soriano (2000), founded on Tijs's construction, is rather restrictive. Indeed, since the conditions on the class of upper-bound games are rather strict, it might be that for certain upper bounds this class is empty.

Here, we explore a more general approach, allowing a larger class of upper-bound games defined in (29) below. It is based on a covariance condition on the selected upper bound which allows the construction of a corresponding lower bound to form a bound pair. This, in turn, allows the construction of a proper compromise value. This method is fully stated in Proposition 5.1, which extends the insight of Sánchez-Soriano (2000, Proposition 3.2).

We recall that a function $f\colon \mathbb{V}^N \to \mathbb{R}^N$ is *translation covariant* if $f(v+x) = f(v) + x$ for any $x \in \mathbb{R}^N$. The next proposition constructs a compromise value from a given translation covariant upper bound.

Let $\eta\colon \mathbb{V}^N \to \mathbb{R}^N$ be translation covariant on $\mathbb{V}^N$. Define

$$\mathbb{B}_u(\eta) = \left\{ v \in \mathbb{V}^N \;\middle|\; v(S) \leqslant \sum_{i \in S} \eta_i(v) \text{ for every } S \subseteq N \right\} \tag{29}$$

as the subclass of *strongly* η*-bound games.*

Proposition 5.1 *Let $\eta\colon \mathbb{V}^N \to \mathbb{R}^N$ be translation covariant on $\mathbb{V}^N$ and $\mathbb{B}_u(\eta)$ the corresponding class of strongly η-bound games.*
Then the function $\mu^\eta\colon \mathbb{B}_u(\eta) \to \mathbb{R}^N$ defined by

$$\mu_i^\eta(v) = \max_{S \subseteq N} R_i(S, v) \qquad \textit{where } R_i(S,v) = v(S) - \sum_{j \in S - i} \eta_j(v) \tag{30}$$

forms a bound pair with η such that μ^η is the lower bound and η is the upper bound over the corresponding class $\mathbb{B}_u(\eta) \subset \mathbb{V}^N$.

Proof. Let $\eta\colon \mathbb{V}^N \to \mathbb{R}^N$ and $\mathbb{B}_u(\eta)$ be defined as formulated in the assertion. If $\mathbb{B}_u(\eta) = \varnothing$ then the assertion of Proposition 5.1 holds trivially.

Assuming $\mathbb{B}_u(\eta) \neq \varnothing$, take any $v \in \mathbb{B}_u(\eta)$. Note that by construction $v(N) \leqslant \sum_{j \in N} \eta_j(v)$. Next, let $\mu^\eta(v)$ be as defined in (30).

First, note that $\mu_i^\eta(v) \geqslant v_i$ for every $v \in \mathbb{B}_u(\eta)$ and player $i \in N$ since $R_i(i, v) = v_i$.

To show that $\mu^\eta(v) \leqslant \eta(v)$, assume to the contrary that there is some $i \in N$ with $\mu_i^\eta(v) > \eta_i(v)$. Hence, from (30), there exists some coalition $S_i \subseteq N$ with $i \in S_i$ and

$$\mu_i^\eta(v) = R_i(S_i, v) = v(S_i) - \sum_{j \in S_i - i} \eta_j(v) > \eta_i(v).$$

But then it follows that $\sum_{j \in S_i} \eta_j(v) < v(S_i)$, which contradicts that $v \in \mathbb{B}_u(\eta)$. Hence, $\mu^\eta(v) \leqslant \eta(v)$, showing that condition (i) of Definition 3.1 is satisfied.

From Sánchez-Soriano (2000, Proposition 3.2), it immediately follows that μ^η defined in (30) is translation covariant on $\mathbb{V}^N$. Hence, in particular this implies that $\mu^\eta(v - \mu^\eta(v)) = 0$, showing that condition (ii-a) of Definition 3.1 is satisfied.

Finally, it follows immediately from covariance of η and the definition of μ^η that condition (ii-b) of Definition 3.1 is satisfied. Therefore, (μ^η, η) forms a proper bound pair on the subclass of upper bound games $\mathbb{B}_u(\eta) \subset \mathbb{V}^N$. ■

Based on the proposition we introduce the proper class of (μ^η, η)-bounded games by

$$\overline{\mathbb{B}}_u(\eta) = \left\{ v \in \mathbb{B}_u(\eta) \;\middle|\; \sum_{i \in N} \mu_i^\eta(v) \leqslant v(N) \leqslant \sum_{i \in N} \eta_i(v) \right\} \tag{31}$$

Proposition 5.1 introduces implicitly a compromise value that is founded on a translation covariant upper bound η. We refer to the corresponding (μ^η, η)-compromise value on $\overline{\mathbb{B}}_u(\eta) \subseteq \mathbb{B}_u(\eta)$ as the *η-Upper Bound Compromise* (η-UBC) value. We note the difference with Proposition 4.3 where a LBC value is defined on $\mathbb{B}_\ell(\mu)$, while an UBC value is defined on a possibly strict subset $\overline{\mathbb{B}}_u(\eta) \subseteq \mathbb{B}_u(\eta) \subset \mathbb{V}^N$.

We explore two natural upper bounds that satisfy the conditions imposed in Proposition 5.1. The first refers to the marginal contributions M of the players to the grand coalition N. This is used in the definition of the τ-value as studied in Tijs (1981, 1987); Tijs and Driessen (1987); Casas-Méndez et al. (2003) and Yanovskaya (2010).

The second natural upper bound concerns the residual that remains for a player $i \in N$ if all other players $j \neq i$ are paid their individual worth v_j. We show that for a certain class of games, the resulting compromise value from this natural upper bound is the CIS-value. With the insight of Section 4.3.2, this leads to the insight that the CIS-value is a compromise value that can be constructed from a lower as well as an upper bound.

A characterisation of UBC values. Sánchez-Soriano (2000) provided a characterisation of the compromise value constructed from a translation covariant upper bound. For completeness, we provide this characterisation here as a restatement.

Lemma 5.2 **(Sánchez-Soriano, 2000, Theorem 3.5)**
Let $\eta \colon \mathbb{V}^N \to \mathbb{R}^N$ be a translation covariant upper bound on $\mathbb{V}^N$ and let μ^η be the constructed lower bound from (30). Then the η-UBC value is the unique value $f \colon \overline{\mathbb{B}}_u(\eta) \to \mathbb{R}^N$ that satisfies the following two properties:

(i) ***Covariance property:***
For every $v \in \overline{\mathbb{B}}_u(\eta)$ and every $x \in \mathbb{R}^N \colon f(\lambda v + x) = \lambda f(v) + x$ for any $\lambda > 0$

(ii) ***Restricted proportionality:***
For every game $v \in \overline{\mathbb{B}}_u(\eta)$ with $\mu^\eta(v) = 0$, there is some $\lambda_v \in \mathbb{R}$ such that $f(v) = \lambda_v \, \eta(v)$.

Note that the characterisation given in Theorem 5.2 deviates from the general axiomatisation of compromise values (Theorem 3.10), which is based on replacing a minimal rights hypothesis with a covariance property.

5.2 Two examples of UBC values

The τ-value, discussed in Section 2 of this paper, is the quintessential example of an UBC value. Here, we discuss shortly two further well-known compromise values that can be constructed from a well-defined upper bound.

5.2.1 The χ-value

Bergantiños and Massó (1996) introduced the notion of the χ-value as an explicit modification of the τ-value through the selection of the Milnor upper bound $\overline{M}$ (Milnor, 1952) instead of the marginal contributions vector. Recalling that for any $v \in \mathbb{V}^N$ and player $i \in N \colon \overline{M}_i(v) = \max_{S \colon i \in S} \left(v(S) - v(S-i) \right)$, we immediately conclude that $\mathbb{B}_u(\overline{M}) = \mathbb{V}^N$. Defining $\mu^{\overline{M}}$ through (30) Bergantiños and Massó (1996) showed that

$$\overline{\mathbb{B}}_u(\overline{M}) = \left\{ v \in \mathbb{V}^N \;\middle|\; \sum_{i \in N} v_i \leqslant v(N) \right\}$$

is the class of *weakly essential* games. The χ-value is now defined as the corresponding $\overline{M}$-UBC value on $\overline{\mathbb{B}}_u(\overline{M})$, given by $\chi = \gamma\left(\cdot ; \mu^{\overline{M}}, \overline{M} \right)$.

Noting that any convex game is always weakly essential, the following insight for convex games follows immediately from the proof of Proposition 3.12 and the definition of the χ-value.

Corollary 5.3 *For every convex game $v \in \mathbb{V}^N_C \colon \chi(v) = \tau(v) = \kappa(v)$.*

5.2.2 The CIS-value redux

In Section 4.2.2, we developed the CIS-value as an LBC-compromise value from the well-defined lower bound $\underline{v}$, where $\underline{v}(v) = (v_1, \ldots, v_n)$ for all $v \in \mathbb{V}^N$. The resulting compromise value based on the construction method set out in Proposition 4.3 was the $(\underline{v}, \eta')$-compromise value, where $\eta'_i(v) = v(N) - \sum_{j \neq i} v_j$ for $v \in \mathbb{V}^N$ and $i \in N$. We concluded there that the $(\underline{v}, \eta')$-compromise value is the CIS-value on $\mathbb{B}_\ell(\underline{v})$ defined by (26).

Using the construction method set out in Proposition 5.1 based on the upper bound η', we can construct the corresponding $(\underline{v}, \eta')$-compromise value as a UBC-compromise value. We show that for a substantial class of games, the $(\underline{v}, \eta')$-compromise value is equal to the CIS-value. However, the different construction method implies that this class of games is smaller than identified for the construction of the CIS-value from the lower bound $\underline{v}$ in Section 4.3.2. For games outside this restricted class, the identified $(\underline{v}, \eta')$-compromise value can be different from the CIS-value and is harder to identify.

Let the upper bound η' be given as above. Then

$$\begin{aligned}
\mathbb{B}_u(\eta') &= \left\{ v \in \mathbb{V}^N \;\middle|\; v(S) \leqslant \sum_{i \in S} \left(v(N) - \sum_{j \in N-i} v_j \right) \text{ for every } S \subseteq N \right\} \\
&= \left\{ v \in V^N \;\middle|\; v(S) \leqslant |S| v(N) - \sum_{i \in S} \sum_{j \in N-i} v_j \text{ for every } S \subseteq N \right\} \\
&= \left\{ v \in V^N \;\middle|\; v(S) \leqslant |S| v(N) - \sum_{i \in N \setminus S} |S| v_i - \sum_{i \in S} (|S|-1) v_i \text{ for every } S \subseteq N \right\} \\
&= \left\{ v \in \mathbb{V}^N \;\middle|\; v(S) - \sum_{i \in S} v_i \leqslant |S| \left(v(N) - \sum_{j \in N} v_j \right) \text{ for every } S \subseteq N \right\}
\end{aligned}$$

Note that $\varnothing \neq \mathbb{B}_u(\eta') \subset \{ v \in \mathbb{V}^N \mid v(N) \geqslant \sum_{i \in N} v_i \}$. Furthermore, $\mathbb{B}_u(\eta')$ contains all games of which the zero-normalisation is monotone.

Proposition 5.4 *Consider the class of cooperative games given by*

$$\widehat{\mathbb{B}} = \left\{ v \in \mathbb{V}^N \;\middle|\; v(S) - \sum_{i \in S} v_i \leqslant (|S|-1) \left(v(N) - \sum_{j \in N} v_j \right) \textit{ for every } S \subseteq N \right\} \subset \mathbb{B}_u(\eta'). \quad (32)$$

Then for every $v \in \widehat{\mathbb{B}}$ the corresponding lower bound defined in Proposition 5.1 is $\underline{v}(v) = (v_1, \ldots, v_n)$. Consequently, the constructed compromise value on $\widehat{\mathbb{B}}$ as asserted in Proposition 5.1 is the corresponding $(\underline{v}, \eta')$-compromise value, being the CIS-value.

Proof. Consider the construction of the lower bound given in (30) for the upper bound η'. Then we compute for $S \subseteq N$ and $i \in S$:

$$\begin{aligned}
R_i(S, v) &= v(S) - \sum_{j \in S-i} \eta'_j(v) = v(S) - \left[(|S|-1) v(N) - \sum_{j \in S-i} \sum_{h \neq j} v_h \right] \\
&= v(S) - (|S|-1) \left(v(N) - \sum_{j \in N} v_j \right) - \sum_{j \in S-i} v_j \\
&= \left(v(S) - \sum_{j \in S} v_j \right) - (|S|-1) \left(v(N) - \sum_{j \in N} v_j \right) + v_i
\end{aligned}$$

Hence, $R_i(S, v) \leqslant v_i$ if and only if $\left(v(S) - \sum_{j\in S} v_j\right) \leqslant (|S| - 1)\ \left(v(N) - \sum_{j\in N} v_j\right)$. By Definition of $\hat{\mathbb{B}}$, this implies that for all $i \in N$: $R_i(S, v) \leqslant v_i$ for all $S \subseteq N$. Therefore, the constructed lower bound is given by $\mu(v) = \underline{v}(v) = (v_1, \ldots, v_n)$ for $v \in \widehat{\mathbb{B}}$. ■

The claim shown above asserts that the constructed compromise value on the class $\widehat{\mathbb{B}}$ is the CIS-value. As mentioned before, this construction method differs from the one based on the lower bound $\underline{v}$ as discussed in Section 3.1.2, since for games in the subclass $\mathbb{B}_u(\eta') \setminus \widehat{\mathbb{B}}$ the constructed compromise value differs from the CIS-value. The next example constructs such a case.

Example 5.5 Consider $N = \{1, 2, 3\}$. Let $A \in \mathbb{R}$ and let $v \in \mathbb{V}^N$ be given by $v_1 = v_2 = 1$, $v_3 = 2$, $v(12) = A$, $v(13) = v(23) = 6$ and $v(N) = 8$. It can easily be verified that the CIS-value of this game is given by $\text{CIS}(v) = \left(2\frac{1}{3}, 2\frac{1}{3}, 3\frac{1}{3}\right)$ for all $A \geqslant 0$.
Next, we determine that $\eta'(v) = (5, 5, 6)$ for all $A \geqslant 0$. This implies that $v \in \mathbb{B}_u(\eta')$ if and only if $A \leqslant 10$. Furthermore, $v \in \widehat{\mathbb{B}}$ if and only if $A \leqslant 6$, implying that for $A \leqslant 6$ the corresponding constructed compromise value is equal to the CIS-value.
For $6 \leqslant A \leqslant 10$ we can compute that the resulting lower bound as stated in (30) is described by $\mu(v) = (A - 5, A - 5, 2)$. Now, $\mu_1(v) + \mu_2(v) + \mu_3(v) = 2A - 8 \leqslant v(N) = 8$ if and only if $6 \leqslant A \leqslant 8$. In that case the resulting compromise value is the feasible balance between $\mu(v) = (A - 5, A - 5, 2)$ and $\eta'(v) = (5, 5, 6)$ computed as $\gamma = \left(\frac{20-A}{12-A}, \frac{20-A}{12-A}, \frac{56-6A}{12-A}\right)$. We remark that this allocation is the CIS-value for $A = 6$, as expected, while for $A = 8$ the resulting compromise value is given by $\gamma = (3, 3, 2)$. ♦

Hence, we conclude from this that the CIS-value has the special property that it is a compromise value that can be constructed from a lower as well as an upper bound on these properly constructed subclasses of games.

6 Concluding remarks: The EANSC value

The "Egalitarian Allocation of Non-Separable Contributions" value or EANSC-value can be understood as the CIS-value of the dual of any cooperative game. It assigns to every player her marginal contribution and then equally taxes all players for the resulting deficit.

In this section, we delve into the unique nature of the EANSC value as a compromise value for two mutually exclusive bound pairs. We demonstrate that it can be derived from the marginal contribution bound M as both an upper bound and a lower bound. Consequently, the two resulting bound pairs encompass the entire space of TU-games on which the EANSC value is indeed properly defined.

The EANSC value as an upper bound based value The EANSC-value has a special role in relation to UBC values. It is itself not an UBC value, but nevertheless it has an interesting relationship with the marginal contributions vector as an upper bound on its payoffs. This is explored here. In particular, the next proposition introduces an innovative perspective on the EANSC-value.

This proposition shows that, not only is the EANSC-value a compromise value, it is a compromise value for the marginal contribution vector M as an upper bound. Its lower bound is constructed from a different methodology as the one introduced for UBC-compromise values. Indeed, the lower bound of the EANSC value can be identified as a solution to a system of equations. Furthermore, the proposition shows that the EANSC-value is defined as a compromise value on a rather large class of games, which includes all essential games.

Proposition 6.1 *Let* $\widetilde{\mathbb{B}} = \{v \in \mathbb{V}^N \mid v(N) \leqslant \sum_{j \in N} M_j(v)\}$ *be the subclass of straightforwardly* M*-upper bounded cooperative games, where* $M_j(v) = v(N) - v(N-j)$ *is the marginal contribution for player* $j \in N$ *in the game* $v \in \mathbb{V}^N$*.*
Let $\tilde{\mu}\colon \widetilde{\mathbb{B}} \to \mathbb{R}^N$ *be a solution to the system of equations*

$$\sum_{j \neq i} \tilde{\mu}_j(v) = v(N-i) \qquad \text{for } i \in N. \tag{33}$$

Then the EANSC-value defined by

$$\text{EANSC}_i(v) = M_i(v) + \tfrac{1}{n}\left(v(N) - \sum_{j \in N} M_j(v)\right) \qquad \text{for every } v \in \mathbb{V}^N \text{ and } i \in N \tag{34}$$

is the $(\tilde{\mu}, M)$*-compromise value on* $\widetilde{\mathbb{B}}$*.*

Proof. We construct the proof of the assertion through the method set out in Proposition 4.3 based on the introduced lower bound $\tilde{\mu}$ defined implicitly as a solution of (33).
For every $v \in \widetilde{\mathbb{B}}$ and every $i \in N$, let $\tilde{\mu}(v)$ be given by

$$\tilde{\mu}_i(v) = M_i(v) + \tfrac{1}{n-1}\left[v(N) - \sum_{j \in N} M_j(v)\right] \tag{35}$$

We claim that the given $\tilde{\mu}$ is a solution of (33). Indeed, for every $v \in \widetilde{\mathbb{B}}$ and $i \in N$:

$$\sum_{j \neq i} \tilde{\mu}_j(v) = \sum_{j \neq i} M_j(v) + \left(v(N) - \sum_{j \in N} M_j(v)\right) = v(N) - M_i(v) = v(N-i).$$

Next, we note that for any $v \in \mathbb{V}^N$: $\sum_{i \in N} \tilde{\mu}_i(v) \leqslant v(N)$ if and only if

$$\sum_{i \in N}(v(N) - v(N-i)) + \frac{n}{n-1}\left(v(N) - \sum_{j \in N}(v(N) - v(N-j)\right) \leqslant v(N) \qquad \text{if and only if}$$

$$n\,v(N) - \sum_{i \in N} v(N-i) + \frac{n}{n-1}v(N) - \frac{n^2}{n-1}v(N) + \frac{n}{n-1}\sum_{j \in N} v(N-j) \leqslant v(N) \quad \text{if and only if}$$

$$\frac{n(n-1) + n - n^2}{n-1}v(N) + \frac{n-n+1}{n-1}\sum_{i \in N} v(N-i) \leqslant v(N) \qquad \text{if and only if}$$

$$\frac{1}{n-1}\sum_{i \in N} v(N-i) \leq v(N)$$

implying that $\sum_{i \in N} M_i(v) = n\,v(N) - \sum_{i \in N} v(N-i) \geqslant v(N)$, which is equivalent to $v \in \widetilde{\mathbb{B}}$. Hence, $\mathbb{B}_\ell(\tilde{\mu}) = \widetilde{\mathbb{B}}$.

Next we check that $\tilde{\mu}(v - \tilde{\mu}(v)) = 0$. For that we note that for every $v \in \widetilde{\mathbb{B}}$ and $i \in N$:

$$M_i(v - \tilde{\mu}(v)) = M_i(v) - \tilde{\mu}_i(v) \quad \text{and}$$
$$(v - \tilde{\mu}(v))(N) = v(N) - \sum_{j \in N} \tilde{\mu}_j(v)$$

leading to the conclusion that

$$\begin{aligned}(v - \tilde{\mu}(v))(N) - \sum_{j \in N} M_j(v - \tilde{\mu}(v)) &= v(N) - \sum_{j \in N} \tilde{\mu}_j(v) - \sum_{j \in N} M_j(v) + \sum_{j \in N} \tilde{\mu}_j(v) \\ &= v(N) - \sum_{j \in N} M_j(v).\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{\mu}_i(v - \tilde{\mu}(v)) &= M_i(v - \tilde{\mu}(v) + \tfrac{1}{n-1}\left[(v - \tilde{\mu}(v)(N) - \sum_{j \in N} M_j((v - \tilde{\mu}(v))\right] \\ &= M_i(v) - \tilde{\mu}_i(v) + \tfrac{1}{n-1}\left[v(N) - \sum_{j \in N} M_j(v)\right] = \tilde{\mu}_i(v) - \tilde{\mu}_i(v) = 0\end{aligned}$$

This shows that $\tilde{\mu}$ is indeed a regular lower bound and that we can apply the method set out in Proposition 4.3. Hence, we determine that for the formulated lower bound $\tilde{\mu}$ the corresponding upper bound, using (22), is given by

$$\tilde{\eta}_i(v) = v(N) - \sum_{j \neq i} \tilde{\mu}_j(v) = v(N) - \sum_{j \neq i} M_j(v) - \frac{1}{n-1} \sum_{j \neq i} v(N) + \frac{1}{n-1} \sum_{j \neq i} \sum_{h \in N} M_h(v) = M_i(v).$$

Furthermore, the corresponding $(\tilde{\mu}, M)$-compromise value defined on $\mathbb{B}_\ell(\tilde{\mu}) = \widetilde{\mathbb{B}}$ is for every $v \in \mathbb{B}_\ell(\tilde{\mu}) = \widetilde{\mathbb{B}}$ and $i \in N$ given by

$$\begin{aligned}\gamma_i(v; \tilde{\mu}) &= \tilde{\mu}_i(v) + \tfrac{1}{n}\left[v(N) - \sum_{j \in N} \tilde{\mu}_i(v)\right] \\ &= M_i(v) + \tfrac{1}{n-1}\left[v(N) - \sum_{j \in N} M_j(v)\right] + \tfrac{1}{n}\left[v(N) - \tfrac{1}{n-1} \sum_{j \in N} v(N - j)\right] \\ &= M_i(v) + \tfrac{1}{n-1}\left[v(N) - \sum_{j \in N} M_j(v)\right] + \tfrac{1}{n(n-1)}\left[\sum_{j \in N} M_j(v) - v(N)\right] \\ &= M_i(v) + \tfrac{1}{n}\left[v(N) - \sum_{j \in N} M_j(v)\right] = \mathrm{EANSC}_i(v),\end{aligned}$$

where the second equality follows by (33) and the third equality follows from the property that the definition of $M_j(v)$ implies that $\sum_{j \in N} v(N - j) = n\, v(N) - \sum_{j \in N} M_j(v)$.
This shows the assertion of the proposition. ■

The EANSC value as an LBC-value From the definition of the EANSC value it should be immediately clear that the EANSC value corresponds to the LBC-value for the lower bound M (Proposition 4.3).[10] The EANSC value as an LBC-value is, therefore, defined over the class $\mathbb{B}_\ell(M) = \left\{ v \in \mathbb{V}^N \mid v(N) \geqslant \sum_{i \in N} M_i(v) \right\}$. From Proposition 4.3 it follows that the EANSC value is the (M, η^M)-value over $\mathbb{B}_\ell(M)$ for the constructed upper bound given by

$$\eta_i^M(v) = v(N) - \sum_{j \neq i} M_j(v) = \sum_{j \neq i} v(N - j) - (n-2)v(N).$$

We now note that the class of the M-lower bounded games $\mathbb{B}_\ell(M)$ is the complement of the class $\widetilde{\mathbb{B}}$ over which the EANSC value is constructed as the $(\tilde{\mu}, M)$-compromise value. Hence, since $\mathbb{B}_\ell(M) \cup \widetilde{\mathbb{B}} = \mathbb{V}^N$, combining these two characterisations of the EANSC value, we have shown that the EANSC value is a compromise value over the whole space $\mathbb{V}^N$ of TU-games, although for two different bound pairs.

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[10]We note that the marginal contribution function M indeed forms a regular lower bound, since $M(v - M(v)) = M(v) - M(v) = 0$ for all games $v \in \mathbb{V}^N$.

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